جامعة بنها

كلية الحاسبات والذكاء الإصطناعي

Probability and Statistics

Lecture 02

Dr. Ahmed Hagag

Faculty of Computers and Artificial Intelligence
Benha University

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Chapter 1: Probability

- Sample Space.
- Events.
- Counting Techniques.
- Probability of an Event.
- Additive Rules.
- Conditional Probability.
- Independence, and the Product Rule.
- Bayes' Rule.

Counting Techniques (1/31)

Introduction (1/2):

In more complicated examples, determining the outcomes in the sample space (or an event) becomes more difficult. In these cases, counts of the numbers of outcomes in the sample space and various events are used to analyze the random experiments.

Introduction (2/2):

Suppose that a **password** on a computer system consists of **eight** characters. Each of these characters must be a **digit** or a **letter** of the alphabet. Each password must contain **at least one digit**.

How many such passwords are there?!

Counting Techniques (2/31)

Multiplication (Product) Rule:

Suppose that a procedure can be **broken down** into a **sequence** of **two tasks**. If there are $\boldsymbol{n_1}$ ways to do the first task and for each of these ways of doing the first task, there are $\boldsymbol{n_2}$ ways to do the second task, then there are $\boldsymbol{n_1 n_2}$ ways to do the procedure and so forth.

The total number of ways to complete the operation is

$$n_1 \times n_2 \times \cdots \times n_k$$

Counting Techniques (3/31)

Example1:

How many sample points are there in the sample space when a pair of dice is thrown once?

Counting Techniques (4/31)

Example1:

How many sample points are there in the sample space when a pair of dice is thrown once?

Solution: The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

Counting Techniques (5/31)

Example2:

The design for a Website is to consist of *four colors*, *three fonts*, and *three positions for an image*.

How many different designs are possible?

Counting Techniques (6/31)

Example2:

The design for a Website is to consist of *four colors*, *three fonts*, and *three positions for an image*.

How many different designs are possible?

Solution: From the multiplication rule, $4 \times 3 \times 3 = 36$ different designs are possible.

Counting Techniques (7/31)

Example3:

In how many different ways can a true-false test consisting of 10 questions be answered?

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In how many different ways can a true-false test consisting of 10 questions be answered?

Solution: Each of the 10 questions can be chosen in two ways, because each question is either true or false. Therefore, the product rule shows there are:

$$2 \times 2 \times \cdots \times 2 = 2^{10} = 1024 \text{ ways to answer the test.}$$

Counting Techniques (9/31)

Example4:

How many different bit strings of *length three* are there?

Counting Techniques (10/31)

Example4:

How many different bit strings of *length three* are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^3 = 8$ different bit strings of length three.

$$2 \times 2 \times 2 = 8$$

Counting Techniques (11/31)

Example5:

How many bit strings of length 5, start and end with 1's?

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Solution:

	Bit #5	Bit #4	Bit #3	Bit #2	Bit #1
Prob.	1	0, 1	0, 1	0, 1	1
Count	1	2	2	2	1

There are $(1 \cdot 2 \cdot 2 \cdot 2 \cdot 1) = 8$ bit strings of length 5, start and end with 1's

Example6:

How many different license plates are available if each plate contains a sequence of *three letters* followed by *three digits*.

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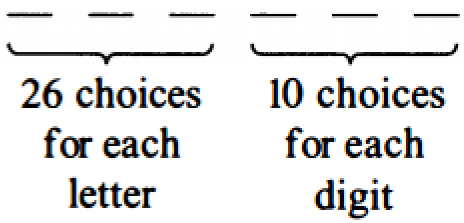

Solution:

There are 26 choices for each of the three letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.

Example7:

If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

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If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

Solution: For the chair position, there are 22 total possibilities. For each of those 22 possibilities, there are 21 possibilities to elect the treasurer. Using the multiplication rule, we obtain $n_1 \times n_2 = 22 \times 21 = 462$ different ways.

Permutations (1/3):

Another useful calculation finds the number of ordered sequences of the elements of a set. Consider a set of elements, such as $S = \{a, b, c\}$.

A permutation of the elements is an ordered sequence of the elements. For example, abc, acb, bac, bca, cab, and cba are all of the permutations of the elements of S.

$$3 \times 2 \times 1 = 6$$

Permutations (2/3):

The number of **permutations** of n different elements is $n!$ where

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Ex. The number of permutations of the four letters a, b, c, and d will be $4! = 24$.

Permutations (3/3):

In some situations, we are interested in the number of arrangements of only some of the elements of a set.

The number of permutations of subsets of ***r*** elements selected from a set of ***n*** different elements is

$$P_r^n = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example1:

Consider a set of elements, such as $S = \{a, b, c, d, e\}$.

What is the number of permutation of subsets of **3** elements selected from S is?

Counting Techniques (19/31)

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Consider a set of elements, such as $S = \{a, b, c, d, e\}$.

What is the number of permutation of subsets of **3** elements selected from S is?

Solution: $\boldsymbol{r = 3}, \;\; \boldsymbol{n = 5}$

$$P_r^n = {}_nP_r = \frac{n!}{(n-r)!} = \frac{5!}{(2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

Counting Techniques (20/31)

Example2:

In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

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In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Solution: $\boldsymbol{r = 3}, \ \boldsymbol{n = 25}$

$$P_r^n = {}_nP_r = \frac{n!}{(n-r)!} = \frac{25!}{(22)!}$$

$$= \frac{25 \times 24 \times \cdots \times 3 \times 2 \times 1}{22 \times 21 \times \cdots \times 2 \times 1} = 13,800$$

Permutations of Similar Objects:

The number of permutations of $n = n_1 + n_2 + \cdots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, … , and n_r are of an rth type is

$$\frac{n!}{n_1!n_2!n_3!\ldots n_r!}$$

Example3:

In a Statistics class, the teacher needs to have 20 students standing in a row. Among these 20 students, there are 12 boy, and 8 girl. How many different ways can they be arranged in a row if only their class level will be distinguished?

Counting Techniques (24/31)

Example3:

In a Statistics class, the teacher needs to have 20 students standing in a row. Among these 20 students, there are 12 boy, and 8 girl. How many different ways can they be arranged in a row if only their class level will be distinguished?

Solution: $\boldsymbol{n = 20}$, $\boldsymbol{n_1 = 12}$, $\boldsymbol{n_2 = 8}$

$$= \frac{n!}{n_1!\, n_2!} = \frac{20!}{12!\, 8!} = 125{,}970$$

Combinations (1/2):

Another counting problem of interest is the number of subsets of r elements that can be selected from a set of n elements. Here, order is not important. These are called *combinations*.

Counting Techniques (25/31)

Combinations (2/2):

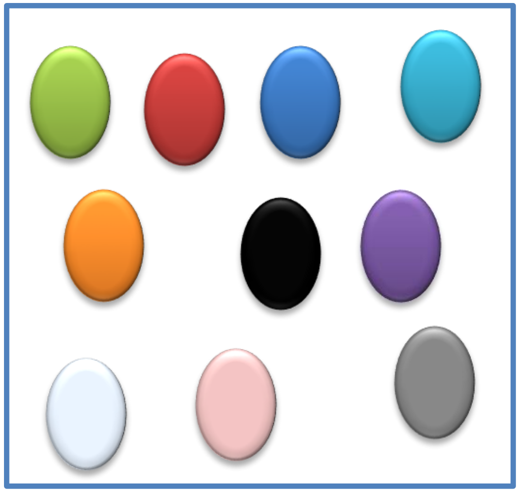
The number of combinations, subsets of r elements that can be selected from a set of n elements, is denoted as $\binom{n}{r}$ or C_r^n and

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Counting Techniques (26/31)

Example1:

How many possible selections of 3 balls from a box contains 10 colored balls?

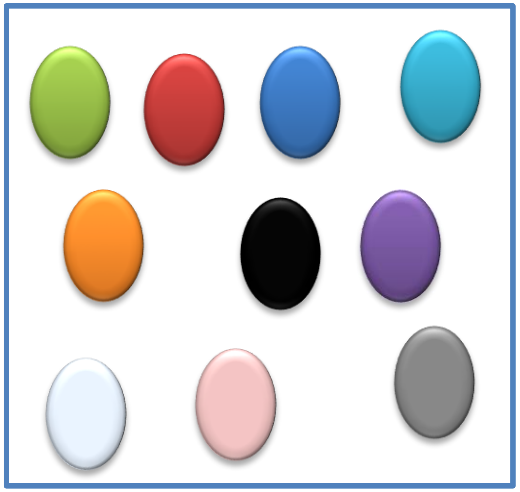


Counting Techniques (27/31)

Example1:

How many possible selections of 3 balls from a box contains 10 colored balls?

Solution: $n = 10 , \quad r = 3$



$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{10!}{3!\,7!} = 120$$

Counting Techniques (28/31)

Example2:

How many ways are there to select 3 candidates from 7 equally qualified recent graduates for work?

Example2:

How many ways are there to select 3 candidates from 7 equally qualified recent graduates for work?

Solution: $\boldsymbol{n = 7\ ,\quad r = 3}$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{7!}{3!\,4!} = 35$$

Counting Techniques (30/31)

Example3:

A bin of 50 manufactured parts. A sample of 6 parts is selected without replacement. That is, each part can be selected only once. How many different samples are there of size 6?

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A bin of 50 manufactured parts. A sample of 6 parts is selected without replacement. That is, each part can be selected only once. How many different samples are there of size 6?

Solution: $\boldsymbol{n = 50}\,,\quad \boldsymbol{r = 6}$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{50!}{6!\,44!} = 15{,}890{,}700$$

Probability of an Event (1/20)

Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur.

"The chance of rain today is 30%" is a statement that quantifies our feeling about the possibility of rain.

Probability of an Event (2/20)

The likelihood of an outcome is quantified by assigning a number from the interval [0, 1] to the outcome (or a percentage from 0 to 100%).

Higher numbers indicate that the outcome is more likely than lower numbers. A 0 indicates an outcome will not occur. A probability of 1 indicates that an outcome will occur with certainty.

Probability of an Event (3/20)

Equally Likely Outcomes:

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

Probability of an Event (3/20)

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Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

Ex. $S = \{1, 2, 3, 4, 5, 6\}$

$$P(1) = \frac{1}{6}, \qquad P(2) = \frac{1}{6}, \qquad \ldots \qquad , \qquad P(6) = \frac{1}{6}$$

Probability of an Event (4/20)

Equally Likely Outcomes:

- S Is a sample space, E is an event
- $E \subseteq S$

$$P(E) = \frac{\#\text{of outcomes in event}(E)}{\text{Total \# of outcomes in sample space}(S)} = \frac{n(E)}{n(S)}$$

$$1 \geq P(E) \geq 0$$

Probability of an Event (5/20)

Example1:

A dice is rolled once. What is the probability of the Event that contains a prime number from the sample space?

Probability of an Event (6/20)

Example1:

A dice is rolled once. What is the probability of the Event that contains a prime number from the sample space?

Solution:

$S = \{1,2,3,4,5,6\}$

$E = \{2,3,5\}$

$P(E) = \frac{3}{6} = 0.5$

Probability of an Event (7/20)

Example2:

A coin is tossed twice. What is the probability that at least 1 head occurs?

Probability of an Event (8/20)

Example2:

A coin is tossed twice. What is the probability that at least 1 head occurs?

Solution:

$S = \{HH, HT, TH, TT\}$

$E = \{HH, HT, TH\}$

$P(E) = \frac{3}{4}$

Probability of an Event (9/20)

NOT Equally Likely Outcomes:

A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$. Then,

$$P(A) = 0.1 + 0.3 = 0.4$$

$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$

$$P(C) = 0.1$$

Probability of an Event (10/20)

Probability of an Event:

For a discrete sample space, the probability of an event E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

Probability of an Event (11/20)

Example3:

A dice is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the dice, find $P(E)$.

Probability of an Event (12/20)

Example3:

A dice is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the dice, find $P(E)$.

Solution:

We assign a probability of w to each odd number and a probability of $2w$ to each even number. Since the sum of the probabilities must be 1, we have $9w = 1$ or $w = 1/9$.

$S = \{1, 2, 3, 4, 5, 6\}$,

$E = \{1, 2, 3\}$

$$P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$$

Probability of an Event (13/20)

Example4:

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that

(a) the dictionary is selected?

(b) 2 novels and 1 book of mathematics are selected?

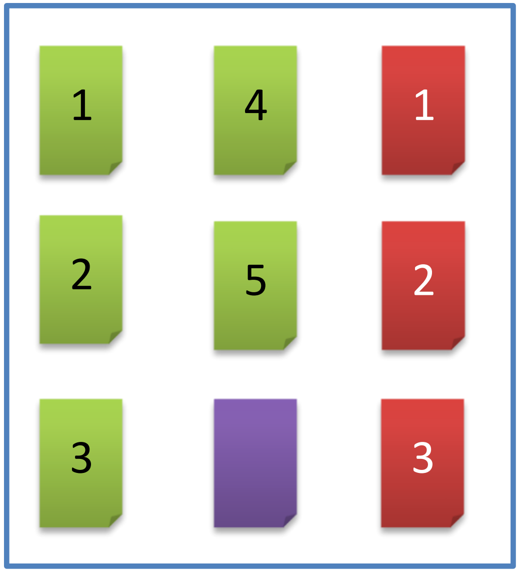
Probability of an Event (14/20)

Example4:

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that

Solution:

$$\#\text{of members of } S = \binom{9}{3} = \frac{9!}{3!\,6!} = 84,$$



Probability of an Event (15/20)

Example4:

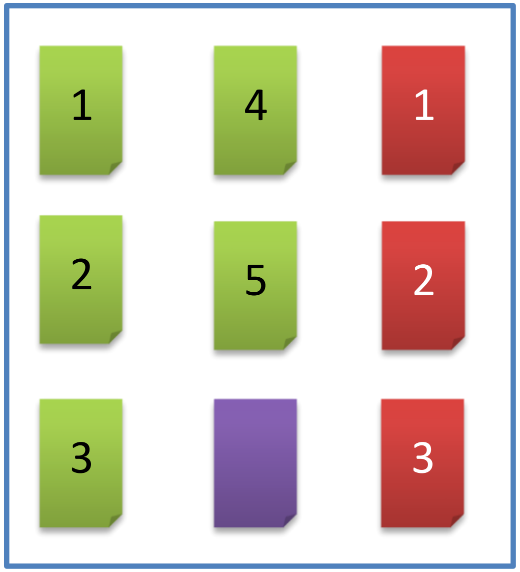
(a) the dictionary is selected?

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that

Solution:

$$\#\text{of members of } S = \binom{9}{3} = \frac{9!}{3!\,6!} = 84,$$

$$\#\text{of members of } E = \binom{1}{1}\binom{8}{2}$$



Probability of an Event (16/20)

Example4:

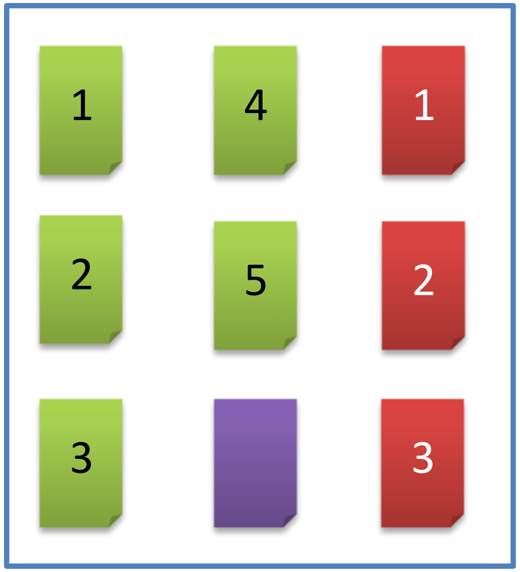
(a) the dictionary is selected?

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that

Solution:

$$\#\text{of members of } E = \binom{1}{1}\binom{8}{2}$$

$$= 1 * \frac{8!}{2!\ 6!} = 28$$

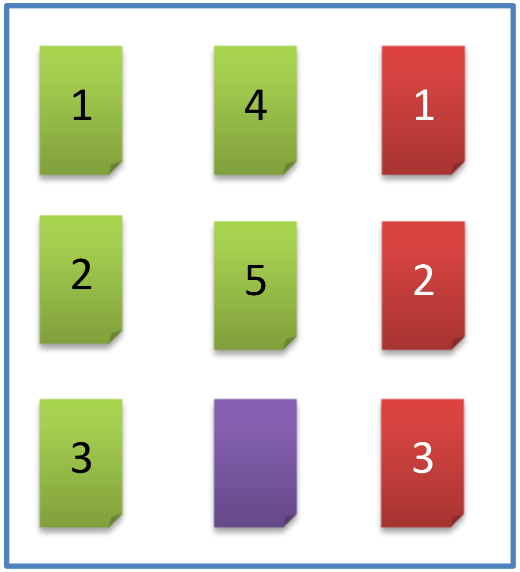


Probability of an Event (17/20)

Example4:

(a) the dictionary is selected?

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that



Solution:

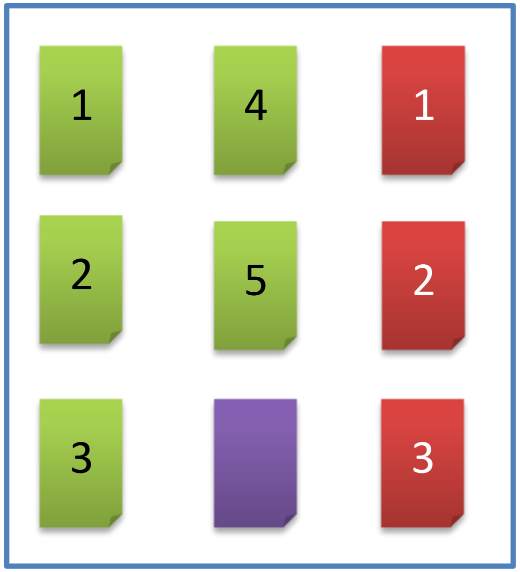
The probability that the dictionary is selected

$$= \frac{28}{84} = \frac{1}{3} = 0.333$$

Probability of an Event (18/20)

Example4: (b) 2 novels and 1 book of mathematics are selected?

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that



Solution:

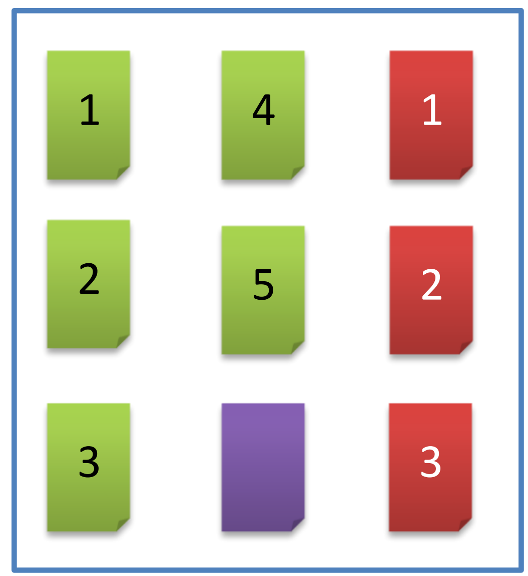
$$\#\text{of members of } E = \binom{5}{2}\binom{3}{1}$$

$$= \frac{5!}{2!\,3!} * \frac{3!}{1!\,2!} = 10 * 3 = 30$$

Probability of an Event (19/20)

Example4: (b) 2 novels and 1 book of mathematics are selected?

If 3 books are picked at random from a box containing 5 novels, 3 books of mathematics, and a dictionary, what is the probability that



<u>Solution:</u>

The probability that 2 novels and

1 book of mathematics are selected

$$= \frac{30}{84} = \frac{5}{14} = 0.357$$

Probability of an Event (20/20)

Axioms of Probability:

S is a sample space, A is an event

$A \subseteq S$

$$\boldsymbol{P(S) = 1}$$

$$\boldsymbol{P(\emptyset) = 0}$$

$$\boldsymbol{0 \leq P(A) \leq 1}$$

$$\boldsymbol{P(A') = 1 - P(A)}$$

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Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #2: https://www.youtube.com/watch?v=NrdCDmSAn7c&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=2

Thank You

Dr. Ahmed Hagag
ahagag@fci.bu.edu.eg